On the statistical-mechanical meaning of the Bousso bound

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The Bousso entropy bound, in its generalized form, is investigated for the case of perfect fluids at local thermodynamic equilibrium and evidence is found that the bound is satisfied if and only if a certain local thermodynamic property holds, emerging when the attempt is made to apply the bound to thin layers of matter. This property consists in the existence of an ultimate lower limit l^* to the thickness of the slices for which a statistical-mechanical description is viable, depending l^* on the thermodynamical variables which define the state of the system locally. This limiting scale, found to be in general much larger than the Planck scale (so that no Planck scale physics must be necessarily invoked to justify it), appears not related to gravity and this suggests that the generalized entropy bound is likely to be rooted on conventional flat-spacetime statistical mechanics, with the maximum admitted entropy being however actually determined also by gravity.

Some examples of ideal fluids are considered in order to identify the mechanisms which can set a lower limit to the statistical-mechanical description and these systems are found to respect the lower limiting scale l^*. The photon gas, in particular, appears to seemingly saturate this limiting scale and the consequence is drawn that for systems consisting of a single slice of a photon gas with thickness l^*, the generalized Bousso bound is saturated. It is argued that this seems to open the way to a peculiar understanding of black hole entropy: if an entropy can meaningfully (i.e. with a second law) be assigned to a black hole, the value $A/4$ for it (where A is the area of the black hole) is required simply by (conventional) statistical mechanics coupled to general relativity.

PACS numbers: 04.20.Cv, 04.70.Dy, 04.62.+v, 05.20.-y, 05.30.-d, 05.70.-a

According to the Bousso covariant entropy bound [1, 2], in a spacetime with Einstein equation and obeying certain energy conditions, the entropy S on a lightsheet L (a lightlike hypersurface spanned by non-expanding light rays emanating orthogonally from an assigned 2-surface and followed until they reach a caustic or a singularity) associated to any given spacelike 2-surface B with area A satisfies

$$S(L) \leq A/4 \tag{1}$$

(in Planck units, the units we will use throughout this paper, except when explicitly stated otherwise). This bound, improving a previous formulation suggested in a cosmological context [3], can be considered a covariant reformulation of the bound [4, 5, 6, 7]

$$S \leq A/4, \tag{2}$$

for the entropy S inside a region with boundary area A, with the aim to overcome some inadequacies of the latter and to obtain full general validity. These bounds are intimately related to the so-called holographic principle [6, 7], in broad terms the conjecture that the physics of any spatial region is completely described by degrees of freedom living on its boundary or also, in the spirit of the work of Bousso, the physics of the lightsheets of any surface B is completely described by degrees of freedom living on B [2].

A generalized form of bound (1) has been proposed in [8]; it states that if (some of) the light rays generating L are terminated on a 2-surface with area A' before reaching a caustic or singularity so that L is now a truncated lightsheet, then

$$S(L) \leq \frac{1}{4}(A - A') \tag{3}$$

and this clearly implies (1) as a particular case. Proofs of the bound in this generalized form can be found in [8] itself and [9, 10], based on the individuation of some very general conditions the material medium should obey, so that, if they hold, the bound can be shown to be satisfied.

In a recent paper [11] another proof has been given, relying on Raychaudhuri equation evolved through local Rindler frames (introduced in [12] as a way to unravel the thermodynamical meaning of Einstein equation; in

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[13] also the relevance of these frames for (horizon) thermodynamic formulation of gravity in terms of the surface term only of Einstein-Hilbert action is stressed). This proof shows (actually for ideal fluids with $\mu \geq 0$, where μ, the chemical potential, includes the possible rest mass of component particles) that bound (3) is satisfied provided a certain inequality holds (in flat spacetime, putting gravity aside), which seems to set a lower limit to the size of the systems for which a meaningful notion of statistical entropy can be given. In particular, for the case $\mu = 0$, the bound (3) is shown to be satisfied *iff* the cited inequality holds, and this would mean that Bousso bound would rest on the existence of an ultimate lower limit to the smallness of the scale at which a thermodynamical description is viable. This claim, we see, in principle goes beyond the finding of a condition, physically plausible, provably sufficient for the validity of the bound; it suggests rather that the bound itself should be the expression, in a general-relativistic context, of some fundamental property of statistical entropy, required, as such, simply by quantum mechanics and the statistical nature of entropy.

The scope of the present work is precisely to investigate if this is the case, namely if the Bousso bound and, we could say, the holographic principle, have some statistical-mechanical flat-spacetime counterpart, on which they could in principle be ultimately rooted.

We start by revisiting the argument of [11]. It was shown in it that, considering a local Rindler horizon through a point p of a classical spacetime satisfying Einstein equation, if δA is the variation of the cross-sectional area A of a pencil of generators of this horizon when an energy flux comes across the horizon, for a homogeneous perfect fluid under local thermodynamic equilibrium conditions Raychaudhuri equation implies

$$-\frac{\delta A}{4} = \pi l \cdot (\rho + p) l A. \tag{4}$$

Here l is the (small) proper length covered by the generators in the fluid rest frame and ρ and p are the local energy density (with possible rest-mass energy included) and pressure, respectively.

On the other hand the entropy dS on the null hypersurface dL spanned by the horizon generators in the same small affine interval is obviously given by $dS = slA$, being s the local entropy density, so that the generalized Bousso bound (3), as applied to dL, is satisfied if and only if

$$s \leq \pi l (\rho + p). \tag{5}$$

This equation turns out to be equation (1.9) of [8], reformulated for thin layers, and also recalls equation (1.11) of the same reference and equation (3.5) of [9]. At the end it looks like a sort of local reformulation of the original bound $S \leq \pi E D$ (here D is the diameter of the smallest sphere circumscribing the system and E is energy) due to Bekenstein [14]. From (5) we get

$$l \geq l^* \equiv \frac{1}{\pi} \, \frac{s}{\rho + p} = \frac{1}{\pi T} \, \left(1 - \frac{\mu n}{\rho + p}\right), \tag{6}$$

where last equality follows from Gibbs-Duhem relation $\rho = Ts - p + \mu n$, being T and n the local temperature and number density respectively. From equation (5) note that Bousso bound demands the null energy condition be satified ($\rho + p \geq 0$, in present case), unphysical negative values for entropy density would be otherwise required. Equation (6), in particular when $\mu = 0$, recalls the expression for the universal bound to the relaxation timescales of perturbed systems found in [15].[1]

At variance with [11], in deriving equations (5) and (6) no restriction is made on μ. The study of the generalized Bousso bound is focused moreover on single slices of matter. We assume that extensivity of entropy means that total entropy on a collection of regions, can be expressed as sum of entropies on each component region, in particular on each component slice of matter, if we have a slice partition. When this is true we say that, for the chosen partition, entropy is extensive on each slice; in this case total entropy cannot depend, of course, on the partition we have performed. If we assume that the entropy which enters the generalized Bousso bound is extensive in this sense, the bound is thus universally valid if and only if it is valid for each of such slices.

In principle this connects to the question of weather it is conceptually possible to have a continuum description of matter entropy so that total entropy can be obtained summing on arbitrarily small sub-regions or, instead, an intrinsic lower limit to the size of the component sub-regions, in our case to the thickness of the slices, must be envisaged. In this perspective, equations (5) or (6) say that if we believe in the generalized

[1] We are grateful to S. Hod for noticing this point.

Bousso bound, then there is an ultimate lower limit to the thickness of the slices for which an extensive notion of entropy can be given, depending this limit on the assigned thermodynamic conditions. Conversely, if for all assigned conditions, the lower limit (6) to the thickness of each slice is supposed to exist, then we are sure the generalized Bousso bound will be universally satisfied.

Let us try to look at this in more detail. Consider a homogeneous macroscopic system M at global thermodynamic equilibrium and with negligible long-range interactions (such as gravity) so that its local thermodynamic description is uniform. Around a point of this system let us imagine to cut a slice of thickness l large enough that we are sure it can be considered macroscopic and the statistical-mechanical description can be applied exactly in the same way as for the whole system. For this slice the notion of entropy is extensive, in the sense, as above, that joining for example two such slices, entropy is doubled. Now imagine repeating this procedure of the cutting of slices, with l going lower and lower, but always requiring the entropy assigned to the slice be extensive. Well, equation (6) says that the generalized Bousso bound (or the holographic principle) obtains if and only if there is an ultimate lower limit l^* to l, determined by the thermodynamical state of M. This argument can evidently be restated in the same way also for systems only at local thermodynamic equilibrium, the thermodynamic observables which set the value of l^* being in this case changing from point to point.

This obviously does not mean that, for every system, l^* is the limiting value l_{min} for which an extensive notion of entropy can be given; if this would be the case, in fact, a slice of every kind of material medium would saturate Bousso bound, provided the thickness is l^*. For most systems we can expect $l_{min} \gg l^*$ so that generalized Bousso bound will be satisfied by far.

As a resumé of what we have said up to now, we can state that the generalized Bousso bound appears to be satisfied if and only if the following property obtains:
"every time a system at local thermodynamic equilibrium has in a point a non-vanishing entropy density, there exists a lower limit l^* to the spatial scale on which an extensive notion of entropy can be furnished. This limit, given by (6), must be understood as an ultimate value, impossible to go beyond; for each given system the actual lower limit l_{min} to the scale for which an extensive notion of entropy can be given will always satisfy

$$l_{min} \geq l^*, \tag{7}$$

with in general $l_{min} \gg l^*$."

This has nothing to do with the mathematical possibility to perform integrals of entropy density treating this latter mathematically as a continuous function; this will be always possible in spite of equation (6) as extensivity down to $l = 0$ can always be mathematically assumed. The point is however that when we require that the *mathematical* entropy in a slice be coincident with the *actual* entropy on that slice as if this slice were alone (the kind of entropy we expect enters Raychaudhuri equation and then equation (4), as the focusing of light rays is due, through Einstein equation, to the *actual* energy-momentum they encounter) things are different and equation (6) appears. In general l_{min} could be expected to be not lower than local thermalization scale. With reference to what we said just now however, the real discriminating point is that the slice be thick enough that the *actual* statistical-mechanical properties we could assign to it when it is taken alone coincide with their *mathematical* value inferred assuming extensivity. This implies that l_{min} values also lower than the local thermalization scale (for example lower than the mean free path for an ordinary gas) could be accepted for consideration, if the area A of the slices can be chosen large enough that a notion of thermodynamic equilibrium can meaningfully be given also for the slice taken alone (for the ordinary gas example, a transverse scale much larger than the mean free path).

We proceed now to study some simple and, in principle, well-understood examples of ideal material media to see what l^* and l_{min} are for them and to verify if condition (7) is satisfied. This will enable us to see which are the mechanisms at work that fix l_{min} values satisfying (7) and then prevent the generalized Bousso bound from being violated. Fluids at (ultra-)relativistic regimes were already considered in [11]. It has been verified there, in particular, that a photon gas should satisfy equation (7), with l^* in its form with $\mu = 0$. A more general argument was also given showing that, for $\mu \geq 0$, for material media with constituent particles with ultra-relativistic dispersion relation, equation (7) should be satisfied. The argument relied on the statement that if a slice has thickness smaller than the quantum spatial uncertainty of the constituent particles, the entropy we could be able to assign to the slice cannot be extensive. This argument seems to imply that the fundamental mechanism at work to protect Bousso bound could simply be quantum mechanics, joined to the macroscopic or statistical nature of entropy.

To investigate this item further, in the present work we consider two non-relativistic systems. The first is a classical ideal non-relativistic Boltzmann gas. In this case one could expect that any argument relying on quantum mechanics should fail, so that for point-like constituent particles no lower limit to l is expected to exist, at least if the saving of extensivity of entropy is concerned. As a consequence one could infer that either this classic system can violate Bousso bound or another mechanism, distinct from quantum mechanics and thus perhaps of statistical nature alone, is also protecting the bound [16].

The equation of state of a non-relativistic ideal Boltzmann gas with constituent particles of mass m, can be expressed in terms of temperature, pressure and chemical potential as

$$\beta p = \frac{z}{\lambda_T^3}, \tag{8}$$

where, when conventional units are chosen, $\beta = 1/kT$, $z \equiv e^{\beta(\mu - mc^2)}$ (it is the fugacity, which turns out to be $z = \lambda_T^3 n$ for a Boltzmann gas) and $\lambda_T \equiv \sqrt{h^2/2\pi mkT}$. In the definition of λ_T, h –in principle a classically undetermined constant with the dimensions of an action, interpreted as the volume of phase space to be assigned to each degree of freedom– is the Planck constant as, with this choice, the calculated statistical entropy describes correctly the experimentally verified entropy of classical gases (Sackur-Tetrode equation for ideal gases at high temperatures) [17]. Even in a purely classical context thus the notions of entropy and chemical potential are in a sense quantum mechanical in that they require the introduction of h. λ_T, the so-called thermal de Broglie wavelength, turns out to be roughly the average de Broglie wavelength for the component particles and this implies that the complete statistical-mechanical description of a classical gas requires the introduction of a sort of "effective size" for the component point-like particles, coincident with their quantum wavelength. The classical nature of the gas is entailed in that $z = \lambda_T^3 n \ll 1$; this assures that the average inter-particle separation be much larger than λ_T. It would desagree with the assumed classical nature of the gas to choose l_{min} significantly lower than λ_T. On the other hand from the very low value of z, the statistical nature of entropy could seem to require $l_{min} \gg \lambda_T$. Given however the fact that in some circumstances the area A of the slices can be chosen in principle even large, in some cases statistics could in principle be recovered even when $l_{min} \equiv \lambda_T$ or lower; this will depend on the actual characteristics of the systems under consideration. We see that this problem with statistics is in the end determined by the request to consider as point-like, particles that actually do have an effective quantum size. Having at disposition systems with infinite extension A could solve the statistics problem, but the quantum mechanical request that l_{min} be not (significantly) lower than λ_T cannot be avoided.

From last equality in (6) and the definition given for z, coming back to Planck units we have

$$l^* = \frac{1}{\pi T}\left(1 - \frac{(m + T ln(\lambda_T^3 n))n}{(m + g\frac{T}{2})n + Tn}\right) \simeq \frac{1}{\pi m}\left(|ln(\lambda_T^3 n)| + \frac{g+2}{2}\right), \tag{9}$$

where g is the number of degrees of freedom per particle and the approximation is justified by the non-relativistic nature of the gas ($\sqrt{T} \ll \sqrt{m}$). If we parametrize the smallness of z as $\lambda_T^3 n = e^{-\chi}$ with $\chi > 0$, from (9) and the definition of λ_T we get

$$\frac{l^*}{\lambda_T} = \frac{1}{\sqrt{2\pi^3}}\sqrt{\frac{T}{m}}\left(\chi + \frac{g+2}{2}\right). \tag{10}$$

Here we see that for χ not too large, that is for not unreasonably high rarefaction conditions, $\chi \ll \sqrt{m/T}$ and thus $l^*/\lambda_T \ll 1$ so that inequality (7) is satisfied.

The lesson we extract from this example is that the ideal classical Boltzmann gas does obey inequality (7) and the generalized Bousso bound, the argument for this appearing in most cases statistical (depending on the size of the actual system under consideration), but, curiously, always quantum mechanical.

The second example we have chosen to consider is a non-relativistic degenerate ideal Fermi gas. We assume then density so high that quantum effects are all important, namely $\lambda_T^3 n \gg 1$, and temperature much smaller than Fermi energy ϵ_F (still non-relativistic). Under these circumstances, number density is such that the statistical requirement alone could be always fulfilled also for $l_{min} = \lambda_T$ or smaller; extensivity of entropy on the other hand gets into trouble for slices with thicknesses significantly smaller than de Broglie wavelength. λ_T, as evident in particular in the limit $T \to 0$, is however no longer a reliable estimator of the average de Broglie wavelength of constituent particles. In any case, calling λ_F the de Broglie wavelength at Fermi energy ($\lambda_F = 2\pi/\sqrt{2m\epsilon_F} = \pi\sqrt{2}/\sqrt{m\epsilon_F}$), under degeneracy conditions l_{min} cannot be significantly lower than λ_F.

Now, as can be inferred from Gibbs-Duhem relation for degenerate Fermi gases (see for example [17]),

$$s = n\frac{\pi^2}{2}\left(\frac{T}{\epsilon_F}\right) \tag{11}$$

at first order in T/ϵ_F, so that from equation (6), neglecting higher order terms in T/ϵ_F, we get

$$l^* = \frac{\pi}{2}\,\frac{T}{\epsilon_F m + \epsilon_F^2}. \tag{12}$$

Finally, using also $\sqrt{\epsilon_F} \ll \sqrt{m}$ (non-relativistic conditions), we have

$$l^* \ll \frac{\pi}{2}\,\frac{1}{m} \ll \pi\sqrt{2}\,\frac{1}{\sqrt{m\epsilon_F}} = \lambda_F \tag{13}$$

and this implies equation (7) is satisfied. In this second example it is thus always the request of the saving of extensivity of entropy that prevents Bousso bound to be violated. From (12), when $T = 0$ we get $l^* = 0$ ($s = 0$) so that the generalized Bousso bound is satisfied trivially.

At the end the question of how l_{min} could precisely be defined as a property of the given material medium (i.e. irrespective of the size of the actual system under consideration; assuming that systems large enough can be considered, to overcome possible problems with statistics) seems to be answered through the following criterium: l_{min} be determined by the requirement that $l_{min} \geq \Delta l$, with Δl the quantum spatial uncertainty of the constituents (or the mimimum between this and their size if they are composite objects). In this regard note that the order-of-magnitude estimate of l_{min} for a photon gas given in [11], suggesting that the inequality $l_{min} \geq l^*$ was broadly respected, gave anyway a value for l_{min} remarkably close to l^*. When, with reference to the definition above for l_{min}, we try to perform a bit more careful estimation we find that l_{min} for a photon gas is like to be practically coincident with l^*. If, in fact, we assume that $\Delta\epsilon = \epsilon/2$ (being $\epsilon = 2.82T$ the peak energy [18]) does roughly capture the maximal uncertainty in photon energy compatible with the given Planck's law at temperature T, from time-energy uncertainty relation $\Delta t \Delta\epsilon \geq \frac{1}{2}$ we obtain $\Delta t \geq \frac{1}{\epsilon}$ so that $\Delta l = \frac{1}{\epsilon} = \frac{1}{2.82T}$. This value is very close to the limiting scale l^*, which in this case is $l^* = \frac{1}{\pi T}$. The entropy of this slice –constructed in such a way that the restriction on the spatial coordinate imposed by slicing has a value with limiting compatibility with the request to not destroy the thermodynamical features of the gas– practically saturates the generalized Bousso bound. We find this a quite strong indication that thermodynamically-meaningful slices of photon gases (for sure, among the most entropic media) can be defined down to scales small enough to be able to saturate the bound, with extensivity of entropy progressively lost when going below.

Let us conclude with some remarks on the results we have obtained. The covariant entropy bound appears to be expression of, or equivalent to, the existence of a fundamental lower limit to the scale of the thermodynamic description. We see that from something living in principle in general relativity we have come to something else linked to concepts of statistical mechanics or thermodynamics on flat spacetime. Inequality (5) corresponds to a limit on entropy for ρ and p assigned. In flat spacetime however this does not mean any intrinsic limit on the total entropy inside an assigned volume, as s can grow without limit with ρ and p. When gravity is turned on, the same limit (5) on entropy density acquires an additional meaning in terms of the gravitational focusing accompanying the given ρ and p and this brings to that now for an assigned geometry, let say on an assigned lightsheet, an absolute intrinsic limit to entropy can be envisaged. A fundamental statistical-mechanical property, the existence of a limiting length, acquires through gravity a fundamental additional valence as absolute limit on entropy.

The role of general relativity is then to shape the peculiar form the generalized Bousso bound has and, in particular, under suitable circumstances the prodigious relation (2). The entropy limit, being determined by gravitational lensing, increases when gravity decreases; without gravity no entropy limit at all could be expected. The setting of this limit however would not be possible if an intrinsic lower limit l^* would not exist to the spatial scale for which a statistical-mechanical description is viable. Gravity succeeds in determining the absolute entropy limit, only thanks to the statistical-mechanical relations (5) and (6).

Equations (5) and (6) do not depend on gravity. In fact, even if our derivation of them hinges on the gravitational lensing of light rays to produce the right cross-sectional area variation required by Einstein equation, also the expression of an assigned entropy as an area depends in the same manner on the strength of gravitational interaction so that this latter does not enter the game. Relations (5) and (6) live in statistical mechanics and should be in principle universally provable through strictly statistical-mechanical arguments; we have shown above that a photon gas seemingly permits to choose slices of limiting thickness small enough to saturate them. The possible (false) impression they could depend on gravity is perhaps generated simply by our path to them, namely we arrived at them from a general relativistic scenario: the generalized Bousso bound. Experimentally-viable gravity theories different from general relativity, should bring to the same estimation (6) for the limiting length.

Note that l^* in (6) can be rewritten in conventional units as

$$l^*_{conv} = \frac{1}{\pi}\,\frac{c\hbar}{k}\,\frac{s}{\rho+p} = \frac{1}{\pi}\,l_{Planck}\,\frac{T_{Planck}}{T}\,\frac{Ts}{\rho+p} \tag{14}$$

being c the speed of light in vacuum, $\hbar$ the (reduced) Planck constant and k the Boltzmann constant and l_{Planck} and T_{Planck} the Planck length and temperature. In the first equality here, the quantum origin of l^* is manifest. The second equality shows that in general $l^*_{conv} \gg l_{Planck}$, so that matter entropy can saturate

Bousso bound on scales much larger than Planck scale, even if usually this is not the case (a recent example is in [19]), just because of the geometry of the lightsheet considered or of the properties of material medium (in general $l_{min} \gg l^*$). On the other hand, as said, thermodynamical conditions can also be such that $l^*_{conv} = 0$ and in this case Bousso bound can never be challenged (the Fermi gas example above). In general putting $\frac{Ts}{\rho+p} \equiv \eta$, for systems for which $\eta = \mathcal{O}(1)$ we obtain $l^*_{conv} = \mathcal{O}(l_{Planck})$ when $T = \mathcal{O}(T_{Planck})$.

In equations (14), l^* does not depend on G, the Newton constant; in particular it retains its value also when $G = 0$. This is expected if this limiting length l^* from which the generalized Bousso bound arises has no relation with gravity and it is then intrinsically statistical mechanical. What does depend on G is the maximum entropy on a lightsheet, namely the actual limiting entropy value entering the generalized Bousso bound, going this maximum as $1/G$.

The generalized Bousso bound implies (provided the ordinary second law is assumed to hold) the generalized second law of thermodynamics [8]. When the areas A and A' entering the generalized Bousso bound are black hole areas, $A/4$ and $A'/4$ are black hole entropies. From our results we see that statistical mechanics and general relativity imply the generalized Bousso bound to hold, with some conventional fluids (the photon gas, for example) seemingly able to saturate it, for thin layers but in any case at a scale very far from the Planck length. We deduce from this that, if to black holes an entropy supposedly entering a second law can be assigned, the fact that its value is $A/4$ appears to be required merely by (conventional) statistical mechanics and general relativity. In other words, always a same thing, black hole entropy, is what is produced using the same ingredients, quantum mechanics plus general relativity; in our case quantum mechanics acts cospiring in order that statistical mechanics respects the lower limiting scale l^*. It is like to have two equivalent ways of looking at black hole entropy: Bekenstein-Hawking approach [14, 20] (black hole entropy from the point of view of the vacuum) and the present statistical-mechanical approach to Bousso bound (black hole entropy from the point of view of material media).

The saturation of Bousso bound, corresponding roughly to about 1 bit of information per Planck area, points, as such, to Planck-scale physics. This limit however, demands for a fundamental property of flat-spacetime statistical mechanics, namely the existence of the lower limiting scale l^*, not affected by Planck-scale physics. This fact seems to agree with the expectation that this limiting value of entropy and, in particular, the bulk of black hole entropy (or of Hawking radiation) are not related to Planck-scale physics (recent results in this direction for the case of acceleration radiation are in [21]).

I thank Jan Geršl for the remarks on an earlier version of this manuscript, in particular for the correction of an error and for an important point he has raised (besides [16]).

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